Pergamon

PII: S0021-8928(96)00098-6

J. Appl. Maths Mechs, Vol. 60, No. 5, pp. 777-788, 1996

0021-8928/96 $24.00+0.00

THE INSTABILITY OF A NON-LINEARLY ELASTIC BEAM UNDER TENSION†

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(*Received* 21 *June* 1995)

The stability of the equilibrium position of a non-linearly elastic rectangular beam under tension when it is subject to small plane perturbations is investigated. The material is assumed to be homogeneous, isotropic and incompressible. Sufficient criteria of stability and instability of the uniform deformation of a beam under tension are obtained. It is established that flexural instability always takes the form of surface bulging. For a thin beam, it is shown that there are no lower-order flexural modes. It is also shown that for medium values of the relative thickness of the beam a loss of stability with the formation of a "neck" occurs for smaller extensions than flexural bulging. The asymptotic form of the critical deformation is constructed with a wide range of applicability. Specific models of highly elastic materials are described for which instability of the equilibrium of the beam under a tensile load is possible.

1. FORMULATION AND SOLUTION OF THE BOUNDARY-VALUE PROBLEM OF THE BIFURCATION OF THE EQUILIBRIUM OF A BEAM

We will consider the homogeneous plane deformation

$$X = \lambda x, \quad Y = \lambda^{-1} y, \quad Z = z \quad (\lambda = \text{const}) \tag{1.1}$$

of an elastic rectilinear beam $|x| \leqslant a$, $|y| \leqslant h$, loaded on the side faces $x = \pm a$ by uniformly distributed normal forces of intensity q (per unit area of the actual configuration). The length and thickness of the beam are assumed to have dimensions of $2a$ and $2h$, respectively. The third dimension—the width—plays no part here and can be chosen arbitrarily. We will assume that there are no mass forces and that the end faces $y = \pm h$ are stress free. The material of the beam is assumed to be homogeneous, isotropic and incompressible. Under these conditions the parameters q and λ are related as follows:

$$\begin{gathered} q = G(\lambda^2 - \lambda^{-2}), \quad G = 2(c_1 + c_2), \quad c_m = \partial\Pi / \partial I_m \quad (m = 1,2) \\ I_1 = \upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2, \quad I_2 = \upsilon_1^{-2} + \upsilon_2^{-2} + \upsilon_3^{-2} \end{gathered} \tag{1.2}$$

Here x, y, z and X, Y, Z are the Cartesian coordinates before and after deformation, respectively, v_l ($l = 1, 2, 3$) are the principal extensions [1, 2], G is the shear modulus of the material for a small deformation from the equilibrium state (1.1) for a simple shear in the XY plane, I_m ($m = 1, 2$) are the first and second principal invariants of the Finger measure of deformation [2], and $\Pi = \Pi(I_1, I_2)$ is the specific potential energy of deformation of the elastic material [1, 2]. The derivatives with respect to I_m ($m = 1, 2$) in (1.2) (and later) are taken for $I_1 = I_2 = \lambda^2 + \lambda^{-2} + 1$. We will assume that the potential Π is a twice continuously differentiable function of the invariants I_1 and I_2 everywhere, with the possible exception of the point where the deformation $I_1 = I_2 = 3$ is measured, and that the following requirements are satisfied:

1. the material satisfies the Hadamard inequality [1, 2] in a certain region U of the space of principal extensions V, containing the point $v_0 = (1, 1, 1)$, corresponding to the undeformed state of the body;
2. at each point of the connected curve L, corresponding to uniform deformation (1.1) and situated wholly in the region U, the inequality $G > 0$ is satisfied.

As a consequence of the fact that the material is incompressible, the space of principal extensions V is a set of points $v = (v_1, v_2, v_3)$ of a three-dimensional arithmetic space R^3 with positive components v_1, v_2, v_3, related by the isochoric condition $v_1 v_2 v_3 = 1$. Note that limitations 1 and 2 on the potential Π do not contradict one another since a consequence of the Hadamard condition (on curve L) is the inequality $G \geqslant 0$ [3, 4].

A violation of the differentiability of the specific energy $\Pi(I_1, I_2)$ at the point $I_1 = I_2 = 3$ is possible, in particular, in materials which possess physical non-linearity even for extremely small deformations

†*Prikl. Mat. Mekh.* Vol. 60, No. 5, pp. 786–798, 1996.

from the uncompressed state. For such materials, the governing relations cannot be linearized in the neighbourhood of the reference configuration.

As an example we can consider a hypothetical material with potential

$$\Pi = d(I_1 - 3)^{\alpha} \quad (d > 0, \alpha \geqslant 1/2) \tag{1.3}$$

for which (1.2) takes the form (C = const > 0)

$$q = 2\alpha d(\lambda + \lambda^{-1})(\lambda - \lambda^{-1})^{2\alpha-1} = C\delta^{2\alpha-1}[1 + O(\delta)] \tag{1.4}$$

Here $\delta \equiv \lambda - 1$ is the principal relative extension of the beam in the stretching direction. It can be seen from (1.4) that when $\alpha \neq 1$, q depends non-linearly on δ for values of δ as small as desired. Nevertheless, when $1/2 \leqslant \alpha < 1$ the derivatives $d\Pi/dI_1$, $d^2\Pi/dI$ are discontinuous at the point $I_1 = I_2 = 3$. Another example is an elastic material with the following potential

$$\Pi = d[1 + (\sqrt{I_1 - 3} - 1)\exp\sqrt{I_1 - 3}\,] \quad (d > 0) \tag{1.5}$$

Unlike the previous case, for model (1.5) the stress q and the extension δ are linearly related as $\delta \to 0$. Nevertheless, the second derivative $d^2\Pi/dI_1^2$ undergoes a discontinuity at the deformation point of reference.
Note that materials (1.3) and (1.5) satisfy requirements 1 and 2 when $U = V$.

We will investigate the stability of the equilibrium configuration (1.1) for small plane perturbations (in the XY plane). The equilibrium equations, linearized in the neighbourhood of the state (1.1), in the case of plane deformation have the form $\lambda \equiv \lambda^{-2}$

$$[(1+\varepsilon)\partial_1^2 + \partial_2^2]w_1 + \lambda^{-1}\partial_1 p = 0 \tag{1.6}$$

$$(\partial_1^2 + \partial_2^2)w_2 + \lambda\partial_2 p = 0$$

$$\gamma\partial_1 w_1 + \partial_2 w_2 = 0 \tag{1.7}$$

$$\varepsilon = 4G^{-1}\lambda^2(1-\gamma^2)^2(c_{11} + 2c_{12} + c_{22}) \tag{1.8}$$

$$c_{lm} = \partial^2\Pi / \partial I_l \partial I_m \quad (l, m = 1, 2)$$

where w_i (i = 1, 2) are the projections of the vector field of small displacement from the configuration (1.1) onto the X, Y axes of a Cartesian system of coordinates, and ∂_i (i = 1, 2) are the operators of differentiation with respect to x and y, respectively. Because the material is incompressible, system (1.6), (1.7) contains an unknown function of the coordinates p, which has the dimensions of pressure and is determined when solving the problem. Note that Eq. (1.7) is also the linearized condition of the material incompressibility.
The components of the linearized Piola stress tensor [1, 2] are expressed in terms of the displacements w^i (i = 1, 2) and the pressure p by the formulae

$$\begin{aligned}
&P_{11} = G[(1+\gamma^2+\varepsilon)\partial_1 w_1 + \lambda^{-1}p], \quad P_{12} = G(\partial_1 w_2 + \gamma\partial_2 w_1) \\
&P_{21} = G(\gamma\partial_1 w_2 + \partial_2 w_1), \quad P_{22} = G(2\partial_2 w_2 + \lambda p) \\
&P_{33} = G[(\varkappa + 2\nu)\partial_1 w_1 + p], \quad P_{i3} = P_{3i} = 0 \quad (i = 1, 2) \\
&\nu = 2G^{-1}c_2\lambda(1-\gamma^2), \quad \varkappa = 4G^{-1}(\lambda + \lambda^{-1})(1-\gamma^2)^2[c_{11} + c_{12}(1+\lambda^2) + c_{22}\lambda^2]
\end{aligned} \tag{1.9}$$

Relations (1.6)–(1.9) are derived using the theory of superposition of a small deformation on a finite deformation [2].
Using (1.9) the linearized boundary conditions on the unloaded end of faces $y = \pm h$ of the beam can be represented in the form

$$(\gamma\partial_1 w_2 + \partial_2 w_1)|_{y=\pm h} = 0, \quad (2\partial_2 w_2 + \lambda p)|_{y=\pm h} = 0 \tag{1.10}$$

On the side faces $x = \pm a$ we assume that the conditions for sliding support [5, 6] are satisfied, i.e.

there are no shear stresses and no normal displacement

$$w_1|_{x=\pm a}=0,\quad (\partial_1 w_2+\gamma\partial_2 w_1)|_{x=\pm a}=0 \tag{1.11}$$

Hence, in the perturbed and unperturbed equilibrium positions of the beam the longitudinal displacements of the particles situated at its ends $x = \pm a$, are the same.

The solution of boundary-value problem (1.6), (1.7), (1.10), (1.11) was obtained previously in [7] and can be written in the form

$$w_1^\pm = W_1^\pm(y)\varphi^\mp(k^\pm x),\quad w_2^\pm = W_2^\pm(y)\varphi^\pm(k^\pm x)$$
$$p^\pm = \pm\lambda P^\pm(y)\varphi^\pm(k^\pm x) \tag{1.12}$$

$$\varphi^+(x)=\cos x,\quad \varphi^-(x)=\sin x$$

$$k^+=\pi m/a,\quad k^-=\pi(2m-1)/(2a)\quad (m=1,2,3\ldots)$$

$$W_1^\pm(y)=\beta[\omega_2(1+\omega_1^2)\Phi_1^\pm(\pm\omega_1k^\pm y)-\omega_1(1+\omega_2^2)\Phi_1^\pm(\pm\omega_2k^\pm y)]$$
$$W_2^\pm(y)=\beta[(\omega_2^2+\gamma^2)\Phi_2^\pm(\pm\omega_1k^\pm y)-(\omega_1^2+\gamma^2)\Phi_2^\pm(\pm\omega_2k^\pm y)] \tag{1.13}$$
$$P^\pm(y)=\pm\beta k^\pm[\omega_2^3(\omega_1^4-1)\Phi_1^\pm(\pm\omega_1k^\pm y)-\omega_1^3(\omega_2^4-1)\Phi_1^\pm(\pm\omega_2k^\pm y)]$$

$$\beta[(\omega_1^2+\gamma^2)^2\omega_2\Phi_1^\mp(\omega_1k^\pm h)-(\omega_2^2+\gamma^2)^2\omega_1\Phi_1^\mp(\omega_2k^\pm h)]=0 \tag{1.14}$$

$$\Phi_1^+(\theta y)=\operatorname{ch}\theta y/\operatorname{sh}\theta h,\quad \Phi_1^-(\theta y)=\operatorname{sh}\theta y/\operatorname{ch}\theta h$$

$$\Phi_2^+(\theta y)=\operatorname{sh}\theta y/\operatorname{sh}\theta h,\quad \Phi_2^-(\theta y)=\operatorname{ch}\theta y/\operatorname{ch}\theta h$$

$$\omega_{1,2}=(\sqrt{\mu+2\gamma}\pm\sqrt{\mu-2\gamma})/2,\quad \mu=1+\gamma^2+\varepsilon,\quad \beta=(\omega_1^2-\omega_2^2)^{-1} \tag{1.15}$$

Here θ is any complex number. In (1.12)–(1.14) there are two rules for matching the "plus" and "minus" signs, which are independent of one another. The first of these only applies to the superscripts of the quantities $w_m^\pm, p^\pm, W_m^\pm, p\pm, \Phi_m^\pm$ $(m = 1, 2)$, where the upper and lower signs correspond to bulging modes, which are symmetric and anti-symmetric with respect to the neutral line $y = 0$ of the beam, respectively. The symmetrical modes describe the process by which a "neck" is formed, while the antisymmetric modes determine the flexural forms of bulging. The second rule operates in the remaining cases; here the choice of upper or lower signs in (1.12)–(1.14) is made consistently. In this rule the upper and lower signs define even and odd bulging modes, that are symmetrical and antisymmetrical with respect to the straight line $x = 0$, respectively. Hence, there are four types of modes—symmetrical even, symmetrical odd, antisymmetrical even and antisymmetrical odd. The nature of the evenness, as will become clear below, is not important.

Expressions (1.13) and (1.14) lose their meaning when $\omega_1 = \omega_2$. In this case we must take the limit in them as $\omega_1 \to \omega_2$.

Note that a consequence of limitation 1 on the potential Π is the inequality

$$\mu + 2\gamma \geqslant 0 \tag{1.16}$$

from which, in particular, it can be seen that the quantities ω_m $(m = 1, 2)$ are either real ($\mu \geqslant 2\gamma$) or complex-conjugate ($|\mu| < 2\gamma$). Moreover, it can be shown that condition (1.16) ensures a monotonic increase in the intensity q of the applied load as the extension λ increases, namely, the inequality $dq/d\lambda > 0$ holds.

The transcendental equations (1.14) determine the critical (or bifurcational) values of the parameter γ, for which the uniform boundary-value problem (1.6), (1.7), (1.10), (1.11) has non-trivial solutions. Henceforth, Eqs (1.14) will be called the characteristic equations.

2. ANALYSIS OF THE CHARACTERISTIC EQUATIONS. CRITERIA OF THE STABILITY AND INSTABILITY OF THE EQUILIBRIUM POSITION

The stability of uniform deformation (1.1) for small plane perturbations has been investigated in detail in [7] in the case of a compressive load ($q < 0$), so we will confine ourselves to be the case of the stretching of a beam ($q > 0$). We will introduce the following notation

$$\tau = h/a, \quad \Gamma = \{\gamma \in (0,1): (\gamma^{-1/2}, \gamma^{1/2}, 1) \in L\}$$

$$L = \{(\upsilon_1, \upsilon_2, \upsilon_3) \in U: \upsilon_1 = \lambda, \ \upsilon_2 = \lambda^{-1}, \ \upsilon_3 = 1 \ (\lambda > 0)\}$$

Theorem 1. If the following inequality is satisfied at each point $\gamma \in \Gamma$

$$\mu + 2\gamma^2 \geqslant 0 \tag{2.1}$$

the characteristic equations (1.14) have no roots, belonging to the set Γ, while the second variation of the potential energy of the beam is positive on any virtual displacement from the equilibrium state (1.1), which indicates that the latter is stable to small plane perturbations for all $\tau > 0$ and $\gamma \in \Gamma$.

Proof. Suppose v is the volume occupied by the body before deformation, ∇ is the nabla-operator in the metric of the undistorted state, $\sigma^{\pm}$ are the side faces $x = \pm a$ of the beam in the actual configuration, $N^{\pm}$ is the current outer normal to the surface $\sigma^{\pm}$, $\mathbf{R}^0$ is the vector of the location of an arbitrary particle in the equilibrium state (1.1), and $\mathbf{R}$ is the radius-vector of the same particle after a certain virtual displacement (parallel to the XY plane) from the equilibrium position (1.1)

$$\mathbf{R} \in C^2(\upsilon), \quad \det \nabla\mathbf{R}|_{\upsilon} = 1, \quad \mathbf{N}^{\pm}(\mathbf{R}^0)\cdot(\mathbf{R} - \mathbf{R}^0)|_{\sigma^{\pm}} = 0 \tag{2.2}$$

The potential energy of the beam [2] for virtual displacement (2.2) has the form

$$W(\mathbf{R}) = \iiint_{\upsilon} \Pi[I_1(\nabla\mathbf{R}\cdot\nabla\mathbf{R}^T), \quad I_2(\nabla\mathbf{R}\cdot\nabla\mathbf{R}^T)]d\upsilon \tag{2.3}$$

The superscript T in (2.3) denotes the operation of transposition of the second-rank tensor. Denoting the variation $\delta\mathbf{R} \equiv \mathbf{R} - \mathbf{R}^0$ by w for brevity and using the technique described in [8], we obtain for the second variation of the functional (2.3) ($\Omega \equiv [-a, a] \times [-h, h]$)

$$\begin{aligned} \delta^2 W &= \frac{1}{2}\iiint_{\upsilon} \mathbf{P}\cdot\cdot\nabla\mathbf{w}^T d\upsilon = \\ &= bG\iint_{\Omega}[(\mu + 2\gamma^2)(\partial_1 w_1)^2 + (\partial_1 w_2)^2 + (\partial_2 w_1)^2 + 2\gamma(\partial_1 w_2)(\partial_2 w_1)]dxdy. \end{aligned} \tag{2.4}$$

Here $2b$ is the width of the beam before deformation and $\mathbf{P}$ is the linearized Piola stress tensor, the components of which are given by (1.9). In deriving (2.4) we used the incompressibility condition (1.7) and the second relation of (1.15).

Since $G > 0$, and when the beam is stretched the parameter γ is less than unity (as follows from (1.2)), when $\mu + 2\gamma^2 > 0$ we see from (2.4) that $\delta^2 W > 0$ for all $\mathbf{w}$, differing from a constant. If $\mu + 2\gamma^2 = 0$, we have $\delta^2 W \geqslant 0$, where the equality is only possible for vector fields $\mathbf{w}$ with components of the form $w_1 = d_1 x + e_1$, $w_2 = d_2 y + e_2$, where d_i and e_i ($i = 1, 2$) are certain constants. But since the displacement field $\mathbf{w}$ necessarily satisfies the incompressibility equation (1.7) and the kinematic limitation $\mathbf{N}^{\pm}(\mathbf{R}^0)\cdot\mathbf{w}|_{\sigma\pm} = 0$ which follows from (2.2), we must have $d_1 = d_2 = e_1 = 0$, i.e. $\mathbf{w} = \text{const}$, which corresponds to a rigid displacement of the body, which is of no interest here.

The theorem is proved.

Note that condition (2.1) lends itself to a simple physical interpretation. We consider the intensity $Q = \lambda^{-1}q$ of the tensile forces applied to the beam when calculating the reference configuration per unit area and we calculate the derivative $dQ/d\lambda$. Using (1.2), (1.8) and (1.15) we obtain

$$dQ/d\lambda = G(\mu + 2\gamma^2) \tag{2.5}$$

Taking into account the fact that $G > 0$, we see from (2.5) that inequality (2.1) is equivalent to the requirement $dQ/d\lambda \geqslant 0$. Also, since the quantity $4hbQ$ is the force acting at the ends $x = \pm a$ of the stretched beam, it follows that so long as the applied force decreases as the extension increases the state of uniform deformation in the stretched beam remains stable. A similar result was obtained in [9] by another method for a rod of compressible material.

Hence, bifurcation of the equilibrium of a beam under tension is only possible on the descending part of the "force-extension" diagram. In practice, a decreasing dependence of the force on the extension can be obtained when the beam is deformed in a rigid testing machine. The limiting conditions of sliding support at the ends of the beam correspond to this case.

The premise of Theorem 1 is met by many well-known models of incompressible elastic materials, for instance by the Treloar, Mooney–Rivlin, Bartenev–Khazanovich and Chernykh–Shubina models [2], to which the following potentials correspond

$$\Pi = d(I_1 - 3) \quad (d > 0) \tag{2.6}$$

$$\Pi = d_1(I_1 - 3) + d_2(I_2 - 3) \quad (d_1, d_2 > 0) \tag{2.7}$$

$$\Pi = d(\upsilon_1 + \upsilon_2 + \upsilon_3 - 3) \quad (d > 0) \tag{2.8}$$

$$\Pi = d_1(\upsilon_1 + \upsilon_2 + \upsilon_3 - 3) + d_2(\upsilon_1^{-1} + \upsilon_2^{-1} + \upsilon_3^{-1} - 3) \quad (d_1, d_2 > 0) \tag{2.9}$$

Note that materials (2.6)–(2.9) satisfy limitations 1 and 2 of Section 1 when $U = V$. Other examples give the following potentials

$$\Pi = d_1 \int \exp[\alpha(I_1 - 3)^2] dI_1 + d_2 \ln(I_2 / 3) \quad (d_1 > 0, d_2 > 0, \alpha > 0) \tag{2.10}$$

$$\Pi = d[(1 + \sigma)(\upsilon_1^\alpha + \upsilon_2^\alpha + \upsilon_3^\alpha - 3) + (1 - \sigma)(\upsilon_1^{-\alpha} + \upsilon_2^{-\alpha} + \upsilon_3^{-\alpha} - 3)] \tag{2.11}$$

$$(d > 0, |\sigma| \leqslant 1, \alpha > 0)$$

$$\Pi = d_1 \int \exp[k_1(I_1 - 3)^{n_1}] dI_1 + d_2 \int \exp[k_2(I_2 - 3)^{n_2}] dI_2 \tag{2.12}$$

$$(d_1 > 0, d_2 > 0, k_1 > 0, k_2 > 0, n_1 > 0, n_2 > 0)$$

which correspond to the Hart-Smith model [10, 11] and the Oden model [12] and to a certain hypothetical material. It can be proved that relation (2.10) satisfies requirements 1 and 2 of Section 1 (where $U = V$) and inequality (2.1) when the following conditions are satisfied

$$\eta < 8, \quad H(\alpha) - \eta / 24 \geqslant 0, \quad \eta \equiv d_2 / d_1 \tag{2.13}$$

$$H(\alpha) \equiv (\sqrt{9\alpha^2 + 2\alpha} - 3\alpha)\exp[(9\alpha + 1 - 3\sqrt{9\alpha^2 + 2\alpha})/2]$$

The same holds for the Oden model (2.11) when $\alpha \geqslant 1$ and for the hypothetical material (2.12) (without additional provisos). Hence, in all the cases considered the "force-extension" diagram is non-decreasing.

Note that when the potential Π is specified in the form of an explicit dependence on the principal extensions v_1, v_2 and v_3, which do not allow a direct transition to be made to the variables I_1 and I_2 (as, for example, in (2.11)), to calculate the parameters G, ν, ε, $\varkappa$ it is more convenient to use the formulae

$$\begin{gathered}
G = \xi\lambda(1 - \gamma^2), \quad \nu = \xi(1 + \gamma)[\Pi_3(1 + \gamma) - \lambda^{-1}(\Pi_1 + \Pi_2)] \\
\varepsilon = \xi[\Pi_2\gamma(\gamma^2 + 3) - \Pi_1(3\gamma^2 + 1)] + G^{-1}(\Pi_{11} - 2\Pi_{12}\gamma + \Pi_{22}\gamma^2) \\
\xi = (\Pi_1 - \Pi_2\gamma)^{-1}, \quad \varkappa = \xi\lambda^{-1}[2(\Pi_1 + \Pi_2) + (\Pi_2 - 2\Pi_3\lambda)(\gamma + 1)^2 + \\
+ (1 - \gamma^2)(\Pi_{22}\lambda^{-1} - \Pi_{12}\lambda + \Pi_{13}\lambda^2 - \Pi_{23})] \\
\Pi_l = \partial\Pi / \partial\upsilon_l, \quad \Pi_{lm} = \partial^2\Pi / \partial\upsilon_l\partial\upsilon_m \quad (l, m = 1, 2, 3)
\end{gathered} \tag{2.14}$$

The derivatives with respect to v_l (l = 1, 2, 3) in relations (2.14) are taken for $v_1 = \lambda$, $v_2 = \lambda^{-1}$, $v_3 = 1$.

We will now consider the case when inequality (2.1) breaks down. We will put

$$\begin{aligned}
&R(\gamma)=\gamma^3-2\gamma^2-\gamma-\mu, \quad S(\gamma)=\gamma^3+2\gamma^2-\gamma+\mu \quad (\gamma\in\Gamma)\\
&\rho(\gamma)=R(\gamma)/S(\gamma) \quad (\gamma\in\Gamma\setminus\Gamma_S^0), \quad \Gamma_S^0=\{\gamma\in\Gamma:\ S(\gamma)=0\}\\
&\Sigma(\gamma)=\mu+2\gamma, \quad \Delta(\gamma)=\mu-2\gamma \quad (\gamma\in\Gamma)\\
&\sigma(\gamma)=\sqrt{\Sigma(\gamma)}, \quad \delta(\gamma)=\sqrt{\Delta(\gamma)}, \quad (\gamma\in\Gamma)\\
&\Gamma_* =\{\gamma\in\Gamma:\ \mu+2\gamma^2<0\}
\end{aligned} \tag{2.15}$$

We will mean by $\sigma(\gamma)$, $\delta(\gamma)$ the branches of the corresponding radicals situated in the closure of the first quadrant. The existence of the latter arises from relation (1.16). Henceforth, so long as it does not lead to misunderstandings, the arguments of the functions $R(\gamma)$, $S(\gamma)$, etc. will be omitted for brevity.

Theorem 2. If the set Γ_* is non-empty, for every element $\gamma\in\Gamma_*$ there will be a value $\tau > 0$ of the relative thickness of the beam for which, at the point γ, a bifurcation of the equilibrium of the uniform deformation (1.1) will occur.

Proof. Using relations (1.15) it can be shown that, with notation (2.15), the characteristic equations (1.14) can be represented in the form

$$R\,\mathrm{sh}\,\sigma k\tau/(\sigma k\tau) = \pm S\,\mathrm{sh}\,\delta k\tau/(\delta k\tau) \tag{2.16}$$
$$k = \pi n/2 \quad (n = 1,\ 2,\ 3,\ldots)$$

where the plus and minus signs denote symmetrical and antisymmetrical bulging modes, while even and odd values of the parameter n denote even and odd bulging modes, respectively.

The conclusion of the theorem is obviously satisfied if and only if, for any fixed value of $\gamma\in\Gamma_*$, at least one of the equations

$$R\,\mathrm{sh}\,\sigma t/(\sigma t) = \pm S\,\mathrm{sh}\,\delta t/(\delta t) \tag{2.17}$$

has positive solutions in t.

In fact, if t_* is a positive root of any of Eqs (2.17), then for $\tau = \tau_* \equiv t_*/k$ the corresponding equation (2.16) is solvable. The converse is also true.

It follows from (2.15) and (1.16) that at each point $\gamma\in\Gamma_*$ the following inequalities are satisfied

$$-2\gamma \leqslant \mu < 2\gamma, \quad |\rho| < 1 \tag{2.18}$$

We will consider two cases. Suppose initially that $|\mu| < 2\gamma$, Then $\sigma > 0$, $\delta = i|\delta|$ (i is the square root of -1), and Eqs (2.17) can be written in the form

$$\sin|\delta|t = \pm\omega\,\mathrm{sh}\,\sigma t, \quad \omega = \rho|\delta|/\sigma \tag{2.19}$$

Clearly, Eq. (2.19) has a denumerable set of positive roots when $\omega = 0$. We will assume that $\omega \neq 0$. To fix our ideas, suppose $\omega > 0$. Then, when choosing the upper sign in Eq. (2.19) the latter is equivalent to the following system of equations

$$\begin{aligned}
&\psi_l(t) = 0 \quad (l = 0,\ \pm1,\ \pm2,\ldots)\\
&\psi_l(t) \equiv |\delta|t - \pi l - (-1)^l \arcsin(\omega\,\mathrm{sh}\,\sigma t)\\
&|t| \leqslant c \equiv \sigma^{-1}\mathrm{arsh}(\omega^{-1})
\end{aligned}$$

where the functions arcsin and arsh are understood in the sense of the principal value. If we calculate the derivatives $\psi_l'(t)$, $\psi_l''(t)$ and note that when $t = c$ the quantities $\psi_{2m}(t)$, $\psi_{2m+1}(t)$ $(m = 0, \pm 1, \pm 2, \ldots)$ take coincident values, then, using the second inequality of (2.18), it can be shown that the product of the functions $\psi_0(t)$, $\psi_1(t)$ vanishes in the interval $(0, c]$. This means that when the upper sign is chosen, Eq. (2.19) necessarily has a positive solution. The case when $\omega < 0$ can be considered in the same way.

Suppose now that $\mu = -2\gamma$. Then $\sigma = 0$, $\delta = 2i\sqrt{\gamma}$, and Eq. (2.17) takes the form

$$\psi(t) \equiv \sin|\delta|t/(|\delta|t) = \pm\rho \tag{2.20}$$

Because the function $\psi(t)$ is continuous, the range of its values on the positive semi-axis covers the section $[0, 1)$, and hence, by virtue of the second inequality in (2.18), at least one of Eqs (2.20) has positive roots. Since this case exhausts possible versions of the relations between μ and 2γ, which follows from the first inequality of (2.18)), the theorem is proved.

A corollary of Theorems 1 *and* 2. In order for the stretched beam to remain stable at each point $\gamma \in \Gamma$ for an arbitrary thickness $\tau > 0$, it is necessary and sufficient that the set $\Gamma_\bullet$ should be empty: $\Gamma_\bullet = \phi$.

Theorem 2 indicates that the limitations imposed on the potential Π in Section 1, do not eliminate the possibility of a bifurcation of the equilibrium of the stretched beam on the falling part of the stress–strain diagram $Q = Q(\lambda)$.

The premise of Theorem 2 is satisfied, for example, for Oden material (2.11) when $\alpha < 1$.

In fact, in this case the Hadamard condition is satisfied in a certain limited region U of space V [4]. Moreover, using (2.14) it can be shown that inequality (2.11) breaks down when $\gamma < \gamma_0$, where $\gamma_0 = [(1-\alpha)/(1+\alpha)]^{1/2} \in \Gamma$, i.e. set $\Gamma_\bullet$ is non-empty.

Other examples of the applicability of Theorem 2 give the following potentials

$$\Pi = d_0(I_2-3)+d_1(I_1-3)+d_2(I_1-3)^2+d_3(I_1-3)^3 \tag{2.21}$$

$$(d_0, d_1, d_2, d_3 = \text{const})$$

$$\Pi = d_0(I_1-3)+d_1(I_2-3)+d_2\ln[1+(I_2-3)/\alpha] \tag{2.22}$$

$$(d_0 > 0, d_1 \geqslant 0, d_2 \geqslant 0, \alpha > 0)$$

$$\Pi = d_0\int\exp[\beta(I_1-3)^2]dI_1+d_1(I_2-3)+d_2\ln[1+(I_2-3)/\alpha] \tag{2.23}$$

$$(d_0 > 0, d_1 \geqslant 0, d_2 \geqslant 0, \alpha > 0, \beta > 0)$$

The first of these corresponds to a Biderman material [13, 11], while the other two correspond to the four-constant and five-constant Alexander models [14, 11]. It has been established that the sufficient conditions for requirements 1 and 2 of Section 1 to be satisfied (when $U = V$) for materials (2.21)–(2.23) have the following respective forms [7, 15]

$$d_0 \geqslant 0, d_1 \geqslant 0, d_3 \geqslant 0, d_1+d_3 > 0, \quad 3d_2+\sqrt{15d_1d_3} \geqslant 0 \tag{2.24}$$

$$d_2 \leqslant 8d_1\alpha \tag{2.25}$$

$$H(\beta) + (8d_1\alpha - d_2)/(8d_0\alpha) \geqslant 0 \tag{2.26}$$

where the function $H(\beta)$ in inequality (2.26) is the same as in (2.13).

Consider the following sets of other constants

$$d_0 = 0, \quad d_1 = 12, \quad d_2 = -25, \quad d_3 = 32 \tag{2.27}$$

$$d_0 = 2, \quad d_1 = 78, \quad d_2 = 8, \quad \alpha = 1/78 \tag{2.28}$$

$$d_0 = 1, \quad d_1 = 54, \quad d_2 = 48, \quad \alpha = 1/9, \quad \beta = 36 \tag{2.29}$$

A direct check shows that each of these satisfies the corresponding constraint (2.24)–(2.26), where in all cases the set $\Gamma_\bullet$ is non-empty.

It should be noted that with a different choice of the elasticity constants, each of the materials (2.21)–(2.23) may turn out to be stable no matter how much they are stretched. In particular, if the quantities d_i ($i = 0, 1, 2, 3$) in (2.11) are non-negative, then for the Biderman model, inequality (2.1) will be satisfied at each point $\gamma \in \Gamma$.

Generally speaking, Theorem 2 does not guarantee that a critical point will exist for any value of the parameter $\tau > 0$. The sufficient conditions for such a point to exist are given below.

Theorem 3. We will assume that the following requirements are satisfied:
1. the set Γ_* is non-empty and contains at least one zero of the function $R(\gamma)$;
2. if $\sup \Gamma_* = 1$, the quantity $\rho(\gamma)$ approaches unity as $\gamma \to 1 - 0$.

The following assertions then hold:
1. for any value of the relative thickness τ both symmetrical and anti-symmetrical bulging are possible;
2. the losses of stability of a bar of any thickness occur for a finite value of the extension λ;
3. a value τ_0 of the parameter τ exists which, for $\tau \leqslant \tau_0$, a symmetrical bifurcation occurs for a shorter extension than antisymmetrical;
4. for any $\tau > 0$ and $n \geqslant 1$, at least one of the modes—symmetrical or anti-symmetrical—necessarily exists.

Here n is the order number of the bulging mode, which is identical with the number of existing nodal lines. Even and odd values of the order number n correspond to even and odd bulging modes, respectively. Consequently, for even n the quantity k^+ plays the part of the "wave-formation" parameter, while for odd n the quantity k^- plays this part. It is easy to show that the integer variable m, which occurs in the definition of the quantities k^+ and k^-, is related to n by the equation $m = \text{entier}[(n + 1)/2]$.

The proof of the theorem is omitted.

A check confirms that all the requirements of Theorem 3 are satisfied, for example, for a Biderman material when $d_0 = 0, d_1 = 27, d_2 = -60, d_3 = 80$ and for the five-constant Alexander model when $d_0 = 1, d_1 = 216, d_2 = 96, \alpha = 1/18, \beta = 36$. Note that these sets of elasticity constants also satisfy inequalities (2.24) and (2.26), respectively, which ensures that requirements 1 and 2 hold when $U = V$.

The results of a numerical calculation of the critical values of the parameter λ for a Biderman material (but just for those values of the constants d_i ($i = 1, 2, 3$) indicated) are given below

τ	0.05	0.10	0.50	1.00	1.50	2.00
λ_1^+	1.17626	1.17634	1.17891	1.18928	-	-
λ_1^-	-	-	-	-	1.19120	1.19145
λ_2^+	1.17634	1.17665	1.18928	-	1.19385	1.20738
λ_2^-	-	-	-	1.19145	-	1.19605
λ_3^+	1.17647	1.17616	-	1.19385	1.20405	1.19858
λ_3^-	-	-	1.19120	-	1.19650	1.19929
λ_4^+	1.17665	1.17791	-	1.20738	1.19858	1.19815
λ_4^-	-	-	1.19145	1.19605	1.19929	1.20005
λ_5^+	1.17688	1.17891	1.19733	1.19790	1.19837	1.19865
λ_5^-	-	-	1.20122	1.20040	1.19966	1.19925
λ_6^+	1.17716	1.18020	1.19385	1.19858	1.19965	1.19899
λ_6^-	-	-	-	1.19929	1.19838	1.19886

The "plus" and "minus" superscripts indicate the critical points for the cases of symmetrical and antisymmetrical bifurcation, respectively, while the numerical subscripts are identical with the order number n of the bulging mode considered. A dash denotes that there is no form of stability loss with a specified number of nodal lines. Note that, for fixed values of $\tau > 0$ and $n \geqslant 1$, each of the characteristic equations (1.14) may be solvable non-uniquely. In such cases, the table shows the critical point closest to the reference origin of the deformation $\lambda = 1$ and which defines the least extension of the beam for which some bulging mode is possible.

An analysis of the numerical results confirms that, for small and medium values of the relative beam thickness (approximately $\tau \leqslant 1$) a "neck" occurs for a smaller extension than the flexural stability loss, and the order number of the corresponding bulging mode is not necessarily equal to unity. Thus, for $\tau = 0.1$ the critical point λ_3^+ is closest to the initial point $\lambda = 1$, which corresponds to the form of stability loss with three nodal lines. For larger values of τ (approximately $\tau > 1$) flexural instability may start prior to the formation of a "neck". This occurs, in particular, when $\tau = 1.5$ and $\tau = 2.0$. Note also that, in the example considered, loss of stability of the stretched beam is observed for a relative extension of the order of 17–20%, which is quite feasible for highly elastic materials.

There is considerable interest in the fact that, under the conditions of Theorem 3, the set Λ_τ of critical values of the parameter λ, corresponding to a fixed thickness $\tau > 0$ for $n = 1, 2, 3, \ldots$, necessarily has at least one point of condensation λ^0 (independently of the choice of τ). Here, as an analysis shows, in the majority of cases the set Λ_τ is localized in a very narrow neighbourhood of λ^0. Hence, the critical values of the relative extension of the beam, corresponding to different bulging modes, are extremely close to one another. For this reason, the actual form of the "neck" in a stretched rod is formed as a result of the superposition of a large number of harmonics of different orders and may differ considerably from sinusoidal.

One other important feature of the bifurcation of the equilibrium of a beam under tension, as can be seen from the above table, is the absence of flexural modes of lower orders for small and medium values of τ. It turns out that this fact is of a universal nature, i.e. it relates to any incompressible material which satisfies requirements 1 and 2 of Section 1.

Theorem 4. If an incompressible elastic material satisfies limitations 1 and 2 of Section 1, then for $n\tau \leqslant 1/3$ there are flexural forms of stability loss of the beam possessing n nodal lines.

The proof will be omitted.

Comparing Theorems 3 and 4 we can conclude that within the limits of the applicability of Theorem 3, for small values of the product $n\tau$ (the conditional limit $n\tau \leqslant 1/3$) only a symmetrical form of stability loss occurs. For medium values of $n\tau$ (approximately $1/3 < n\tau \geqslant 4/3$) antisymmetrical bulging, along with symmetrical bulging, is most often possible, but, as a rule, occurs for a greater extension of the beam than symmetrical. Finally, for large values of the product $n\tau$ ($n\tau > 4/3$), under the conditions of Theorem 3, an alternation of symmetrical and antisymmetrical modes is observed when $n\tau$ increases with no limit.

Note that materials that satisfy the requirements of Theorem 3, possess two characteristic features:

(a) stability losses of the stretched beam occur for all values of $\tau > 0$;

(b) the minimum critical value $\lambda_{\min}(\tau)$ (for a given thickness τ) is a bounded function of the parameter τ on the ray $(0, +\infty)$.

Such materials, by analogy with [7], can be called materials with a low stiffness for stretching. Analysis shows that, as in the case of the compression of a beam, limitations 1 and 2 of Section 1, imposed on the potential Π, allow of the existence of materials with bifurcation properties differing from *a* and *b*.

Examples are the models (2.21)–(2.23) for values of the elasticity constants (2.27)–(2.29), respectively, for which property *a* breaks down. Such materials, which possess a "limiting" thickness, are materials of increased stiffness for stretching. Another example gives the potential

$$\Pi = d_1\Phi(J_1) + d_2\Phi(J_2) \quad (d_1 > 0, d_2 > 0)$$

$$J_m = [I_m - 1 - \sqrt{(I_m - 3)(I_m + 1)}]/2 \quad (m = 1, 2)$$

$$\Phi(x) = \int_x^1 y^{-3/2}\left(\frac{1-y}{1+y}\right)^\beta \exp\left(\frac{1}{2}y\right)dy, \quad x \in (0, 1), \quad \beta \in \left(0, \frac{1}{4}\right)$$

which is subject to requirement *a*, but does not satisfy condition *b*. Following [7], it is logical to regard it as belonging to materials of medium stiffness for stretching. Hence, the problem of classifying materials, investigated in detail in [7] as it applies to the compression of a beam, does not lose its validity in the case of stretching, but will not be considered in detail here (due to lack of space). We will merely note that the bifurcation properties of a material for stretching and compression are, generally speaking, not identical.

3. THE ASYMPTOTIC BEHAVIOUR OF THE CRITICAL VALUES OF THE DEFORMATION AND THE LOAD AS $\tau \to 0$

As follows from Theorem 4, we are dealing with the bifurcation parameters corresponding to symmetrical bulging modes, since for the antisymmetrical modes, by virtue of this theorem, the formulation of the problem has no meaning.

Theorem 5. We will assume that the set Γ_* is non-empty, and $\sup \Gamma_* < 1$. Then, for any order number $n \geqslant 1$ a value $\tau_n > 0$ of the relative thickness τ exists such that for $\tau \leqslant \tau_n$ at least one symmetrical form of stability loss of the beam exists possessing n nodal lines, where, for the corresponding critical value $\gamma_n(\tau)$ of the parameter γ, we have the following asymptotic formula ($\tau \to 0$)

$$\gamma_n(\tau) = \gamma_* - \gamma_1 k_n^2 \tau^2 - \gamma_2 k_n^4 \tau^4 + O(\tau^6) \tag{3.1}$$

$$\gamma_* = \sup \Gamma_*, \quad k_n = \pi n / 2 \quad (n = 1, 2, 3, \ldots)$$

$$\gamma_1 = \gamma_* \zeta_1 / (3\mu_1), \quad \gamma_2 = \gamma_*^2 \zeta_2 / (45\mu_1^3), \quad \zeta_1 = \gamma_*(1 - \gamma_*^2)$$

$$\zeta_2 = \zeta_1 [5\mu_2 \zeta_1 + 10\mu_1(2\gamma_*^2 - 1) + 2\gamma_* \mu_1^2]$$

Here μ_1 and μ_2 are the coefficients of the expansion of the function $M(\gamma) \equiv \mu + 2\gamma^2$ in a Taylor series at the point

$$M(\gamma) = \mu_1(\gamma - \gamma_*) + \mu_2(\gamma - \gamma_*)^2 + \ldots$$

The proof of the theorem will be omitted.

Notes. 1. From a physical point of view, the quantity γ_* corresponds to the first maximum point λ_* on the curve $Q = Q(\lambda)$, which defines the relation between the tension and the extension of the beam.

2. If, in addition, the material satisfies all the requirements of Theorem 3, then, for the above-mentioned thicknesses τ_n ($n = 1, 2, 3, \ldots$) we have the lower limit $\tau_n \geqslant 1/(3n)$.

3. When the bifurcation point is not unique for the specified values of n and τ, we mean by $\gamma_n(\tau)$ the deformation closest to the reference value $\gamma = 1$, i.e. corresponding to the least extension of the beam, for which the form of stability loss with n nodal lines is possible.

4. Taking into account the relation between the parameters γ and λ ($\gamma = \lambda^{-2}$), and also relation (1.2), we can obtain asymptotic formulae for the critical values $\lambda_n(\tau)$ and $q_n(\tau)$, similar in their structure to (3.1). The fact that the critical stress $q_n(\tau)$, unlike the compression case [16, 17], approaches a non-zero limit as $\tau \to 0$, which is identical with the value of q for the maximum point λ_* is of interest.

5. Comparison of the asymptotic form (3.1) with the results of a numerical calculation confirms that it is applicable over a wide range (approximately $0 < \tau \leqslant 1/n$) and has very high accuracy. In particular, for a Biderman material with values of the elasticity constants $d_0 = 0$, $d_1 = 27$, $d_2 = -60$, $d_3 = 80$, the asymptotic representation for the critical extension $\lambda_n(\tau)$, which follows from (3.1), has the form

$$\lambda_n(\tau) = 1.176239 + 0.004066 k_n^2 \tau^2 + 0.000269 k_n^4 \tau^4 + \ldots \tag{3.2}$$

For $n = 1$ the relative error of (3.2) is 0.005% for $\tau = 0.5$, 0.026% for $\tau = 0.7$ and 0.14% for $\tau = 1.0$. Comparing these results with those obtained previously in [16, 17], we conclude that the range of applicability of asymptotic form (3.1) is three to four times wider than the corresponding formulae for the compression of a beam.

4. ANALYSIS OF THE FORMS OF STABILITY LOSS

We will introduce the following notation

$$k = \pi n / 2, \quad \xi = k\tau\sqrt{2\gamma + \mu} / 2, \quad \eta = k\tau\sqrt{2\gamma - \mu} / 2$$

$$\psi^+(\xi) = \operatorname{ch}\xi, \quad \psi^-(\xi) = \operatorname{sh}\xi, \quad K^\pm = K_1^\pm / K_2^\pm$$

$$K_1^\pm = (\mu + 2\gamma^2)\psi^\pm(\xi)\sin\eta + \sqrt{4\gamma^2 - \mu^2}\,\psi^\mp(\xi)\cos\eta \tag{4.1}$$

$$K_2^\pm = (\mu + 2\gamma^2)\psi^\mp(\xi)\cos\eta - \sqrt{4\gamma^2 - \mu^2}\,\psi^\pm(\xi)\sin\eta$$

$$\varphi(\gamma) = \arccos[\gamma(\gamma^2 + 1 + \mu)/|S(\gamma)|] \quad (\gamma \in \Gamma \setminus \Gamma_S^0)$$

In (4.1) we simultaneously take both the upper or lower signs. Note that in the set Γ_* all the radicands are non-negative. For the radicals it is usual to choose the non-negative branches.

Theorem 6. Suppose $\gamma \in \Gamma_*$ is the bifurcation point for the specified value $\tau > 0$ for symmetrical (antisymmetrical) bulging of the stretched beam and n is the order number of the corresponding bulging mode with bowing amplitude $W_2^+(y)$ ($W_2^-(y)$). Then the function $W_2^+(y)$ ($W_2^-(y)$) has a single change of sign over the beam thickness when the condition $2\eta \leqslant \pi + \varphi(\gamma)$ is satisfied and no less than three sign changes when the last condition is violated (there are always no less than two sign changes over the thickness of the beam). The total number of sign changes is given by the formula $N^+ = 2n^+ + 1$ ($N^- = 2n^-$), where

$$n^{\pm} = \operatorname{entier}\left(\frac{l^{\pm}}{2}\right) + \frac{1+(-1)^{l^{\pm}+1}}{2}\operatorname{sign}|K_1^{\pm}K_2^{\pm}|\times$$
$$\times\operatorname{entier}\left[\frac{1}{2}+\frac{1}{2}\operatorname{sign}(\operatorname{tg}\eta - K^{\pm}(\operatorname{th}\xi)^{\pm 1})\right] \tag{4.2}$$
$$l^{\pm} = \operatorname{entier}\left(\frac{2\eta}{\pi}\right) + 1 - \operatorname{sign}|K_2^{\pm}| + \frac{1}{2}[\operatorname{sign}(K_1^{\pm}K_2^{\pm}) + \operatorname{sign}|K_1^{\pm}K_2^{\pm}|]\Lambda^{\pm}$$
$$\Lambda^{+} = \operatorname{sign}(K^{+}\xi - \eta) + \operatorname{sign}|K^{+}\xi - \eta|, \quad \Lambda^{-} = 1$$

It is assumed that at points of discontinuity the function $\operatorname{tg} x$ takes the value $-\infty$.
The proof is omitted.

Corollary. If the inequality $n\tau \leqslant 4/3$ is satisfied, then when a symmetrical form of bulging with n nodal lines exists the bowing amplitude $W_2^+(y)$ changes sign only at the point $y = 0$.

Note that this assertion, like the conclusion of Theorem 4, is universal in the sense that it relates to any incompressible material which satisfies limitations 1 and 2 of Section 1. The assertion that the bowing amplitude $W^-_2(y)$ has a minimum of two sign changes also has the same property.

Formulae (4.1) and (4.2) show that as the product $n\tau$ increases, the parameters $N^{\pm}$, which define the number of sign changes in the bowing amplitudes $W_2^{\pm}(y)$, may increase without limit. This occurs, in particular, in the conditions of Theorem 3. Since $N^- \geqslant 2$ always, for antisymmetric bulging of the stretched beam there is necessarily at least one internal layer which preserves its initial form and dimensions when deformed. The visible deformation of the beam is concentrated in the lower and upper external layers. Hence, the flexural stability loss takes the form of surface bulging. Exactly the same situation occurs in symmetrical bifurcation if $N^+ \geqslant 3$. In the opposite case, the whole region occupied by the body undergoes deformation, and the stability loss has a global character. As the corollary of Theorem 6 shows, such instability occurs for small and medium values of the product $n(\tau)$.

This research was supported by the Russian Foundation for Basic Research (93-013-16497).

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Translated by R.C.G.